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Moment Methods in Two Point Padé Approximation

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In the separable Hilbert space $(H, \langle\cdot,\cdot\rangle)$ the following "operator moment problem" is solved: given a complex sequence $(c_k)_{k\in\mathbf{Z}}$ generated by a meromorphic function f, find $T\in B(H)$ and $u_0\in H$ such that $\langle T^k u_0, u_0\rangle = c_k$ $(k\in\mathbf{Z})$. If the sequence $(c_k)_{k\in\mathbf{Z}}$ is "normal," an adapted form of Vorobyev's method of moments yields a sequence of two point Padé approximants to f. A sufficient condition for convergence of this sequence of approximants is given.

INTRODUCTION AND SUMMARY

In [2] an adapted form of the method of moments of Vorobyev [5] was used to generate a sequence of ordinary Padé approximants and to obtain a convergence result for this sequence.

It was van Rossum who raised the question of whether similar results could be obtained for two point Padé approximants. The present paper answers this question positively.

Given a function f with power series developments $f(z)=\sum_{n=0}^{\infty} c_n z^n$ with $c_0=1$ at 0 and $f(z)=-\sum_{n=1}^{\infty} c_{-n} z^{-n}$ at ∞ we consider the $((n-1)/n;m)$ two point Padé approximants $R_n^{(m)}(z)$ (see Definition 1.1). For $m=n$ we get ordinary Padé approximants. Certain normality conditions for these two point approximants lead to a biorthogonal system in the algebra $\mathscr{A}$ of the Laurent polynomials with respect to the linear functional Ω on $\mathscr{A}$ defined by $\Omega(z^n)=c_n$ $(n\in\mathbf{Z})$.

If f is meromorphic on $\mathbf{C}^*\backslash\{p\}$, $p\neq 0$, $p\neq\infty$, then the operator moment problem for the sequence $(c_n)_{n\in\mathbf{Z}}$ (see Section 2) has a solution $T\colon H\to H$ which is a linear isomorphism in the separable Hilbert space H. By means of the operator T we construct from the biorthogonal system of Laurent polynomials a biorthogonal system in H. Using this biorthogonal system in H and the normality conditions for the sequence $(c_n)_{n\in\mathbf{Z}}$ we get a sequence of

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linear projections $E_n: H \to H$ with finite dimensional range such that the operators $T_n = E_n T E_n$ satisfy

$$\langle (I - zT_n)^{-1} e_0, e_0 \rangle = R_n^{(m)}(z) \qquad \text{for} \quad z \in \mathbf{C}^* \setminus (\{p\} \cup \{\text{poles}\})$$

($e_0 \in H$ fixed, $\|e_0\| = 1$).

If, moreover, the first sequence of the biorthogonal system in H is a Schauder basis of H (in which case the second sequence is also a Schauder basis of H), then $R_n^{(m)}(z) \to f(z)$ as $n \to \infty$ for $z \in \mathbf{C}^* \setminus (\{p\} \cup \{\text{poles}\})$, faster than any geometric progression.

1. Let f be a complex function which is holomorphic at 0 and at ∞. Suppose

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \text{ in some neighborhood of } 0 \tag{1.1}$$

and

$$f(z) = -\sum_{n=1}^{\infty} c_{-n} z^{-n} \text{ in some neighborhood of } \infty \tag{1.2}$$

and assume $c_0 = 1$ and $c_{-1} \neq 0$.

Just as in the case of ordinary Padé approximation one can prove that for each $n \in N$ and each integer m with $-n \leqslant m \leqslant n$ there exists precisely one rational function $R_n^{(m)}(z) = U_{n-1}^{(m)}(z)/V_n^{(m)}(z)$ where $U_{n-1}^{(m)}$ and $V_n^{(m)}$ are polynomials with $\deg U_{n-1}^{(m)} \leqslant n-1$ and $\deg V_n^{(m)} \leqslant n$ such that

$$f(z) - R_n^{(m)}(z) = O(z^{n+m}) \qquad \text{as} \quad z \to 0 \tag{1.3}$$

and

$$f(z) - R_n^{(m)}(z) = O(z^{-n+m-1}) \qquad \text{as} \quad z \to \infty. \tag{1.4}$$

DEFINITION 1.1. The unique rational function $R_n^{(m)}(z) = U_{n-1}^{(m)}(z)/V_n^{(m)}(z)$ where $U_{n-1}^{(m)}$ and $V_n^{(m)}$ are polynomials with $\deg U_{n-1}^{(m)} \leqslant n-1$ and $\deg V_n^{(m)} \leqslant n$ ($n \in N$, $-n \leqslant m \leqslant n$, $m \in \mathbf{Z}$) which satisfies (1.3) and (1.4) is called the $((n-1)/n; m)$ *two point Padé approximant to* f. We say that $R_n^{(m)}$ is *normal* if $R_n^{(m)}$ has exactly one representation

$$R_n^{(m)}(z) = \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} + b_n z^n} \tag{1.5}$$

with $b_0 = 1$ and $b_n \neq 0$ and

$$f(z) - R_n^{(m)}(z) \neq O(z^{n+m+1}) \qquad \text{as} \quad z \to 0$$

and

$$f(z) - R_n^{(m)}(z) \neq O(z^{-n+m-2}) \qquad \text{as} \quad z \to \infty.$$

Given (1.1) and (1.2), relations (1.3) and (1.4) can be written as systems of linear equations in $a_0,\dots, a_{n-1}$ and $b_0,\dots, b_n$ when $R_n^{(m)}$ is of the form (1.5). Elimination of $a_0,\dots, a_{n-1}$ gives

$$\begin{array}{l} c_m \quad b_0 + c_{m-1} \quad b_1 + \cdots + c_{m-n} \quad b_n = 0, \\ c_{m+1} \quad b_0 + c_m \quad b_1 + \cdots + c_{m-n+1} b_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{m+n-1} b_0 + c_{m+n-2} b_1 + \cdots + c_{m-1} \quad b_n = 0 \end{array} \tag{1.6}$$

and it follows easily that normality of $R_n^{(m)}$ is equivalent to

$$H_n^{(m-n)} \neq 0, \qquad H_n^{(m-n+1)} \neq 0, \qquad H_{n+1}^{(m-n)} \neq 0 \qquad \text{and} \qquad H_{n+1}^{(m-n-1)} \neq 0, \tag{1.7}$$

where

$$H_q^{(p)} = \begin{vmatrix} c_p & c_{p+1} \cdots c_{p+q-1} \\ c_{p+1} & c_{p+2} \cdots c_{p+q} \\ \cdots\cdots & \cdots\cdots\cdots \\ c_{p+q-1} & c_{p+q} \cdots c_{p+2q-2} \end{vmatrix} \qquad \text{for} \quad p \in \mathbf{Z} \quad \text{and} \quad q \in \mathbf{N}.$$

DEFINITION 1.2. The sequence $(c_n)_{n\in\mathbf{Z}}$ is *m-normal* for some integer m if (1.7) is valid for each $n \in \mathbf{N}$ and $(c_n)_{n\in\mathbf{Z}}$ is *m-seminormal* if $H_n^{(m-n)} \neq 0$ for each $n \in \mathbf{N}$.

If $(c_n)_{n\in\mathbf{Z}}$ is m-normal, then $R_n^{(m)}$ is normal for each $n \in \mathbf{N}$ such that $n \geqslant |m|$. In the sequel of this section we assume that $(c_n)_{n\in\mathbf{Z}}$ is m-seminormal for some $m \in \mathbf{Z}$.

Then (1.6) has for each $n \in \mathbf{N}$ a unique solution $b_0,\dots, b_n$ with $b_0 = 1$ and for the sequence $(P_n^{(m)})_{n=0}^{\infty}$ of polynomials defined by

$$P_0^{(m)}(z) = 1$$

and

$$P_n^{(m)}(z) = \frac{1}{H_n^{(m-n)}} \begin{vmatrix} c_{m-n} \cdots c_m \\ \cdots\cdots\cdots \\ c_{m-1} \cdots c_{m+n-1} \\ 1 \quad \cdots z^n \end{vmatrix}, \qquad n = 1, 2, \dots,$$

we have $P_n^{(m)}(z)=z^n+b_1z^{n-1}+\cdots+b_{n-1}z+b_n$ so $z^nP_n^{(m)}(z^{-1})$ is just the denominator of $R_n^{(m)}(z)$ for $n\in\mathbf{N}$ and $n\geqslant|m|$. Let $\mathscr{A}$ be the algebra of the Laurent polynomials in z, i.e., the algebra of all functions of the form

$$a_pz^p+a_{p+1}z^{p+1}+\cdots+a_qz^q$$

with $p,q\in\mathbf{Z}$ and $a_p,\ldots,a_q\in\mathbf{C}$, and let Ω be the linear functional on $\mathscr{A}$ defined by

$$\Omega(a_pz^p+\cdots+a_qz^q)=a_pc_p+\cdots+a_qc_q.$$

Then we extend $(P_n^{(m)}(z))_{n=0}^{\infty}$ to a biorthogonal system $\{P_n^{(m)}(z); z^{m-1}Q_n^{(m)}(z)\}_{n=0}^{\infty}$ in $\mathscr{A}$ with respect to Ω if we define

$$Q_0^{(m)}(z)=1$$

and

$$Q_n^{(m)}(z)=\frac{(-1)^n}{H_n^{(m-n)}}\begin{vmatrix} c_{m-n-1} & \cdots & c_{m-1}\\ \cdots & \cdots & \cdots\\ c_{m-2} & \cdots & c_{m+n-2}\\ z^{-n} & \cdots & 1\end{vmatrix},\qquad n=1,2,\ldots.$$

Remark 1.1. In Section 2 we derive from this Ω-biorthogonal system an ordinary biorthogonal system in a Hilbert space, in the same way as the Lanczos biorthogonal system is obtained from an orthogonal system of polynomials.

Remark 1.2. If $g(z)=c_{-1}+zf(z)$, then $g(z)=\sum_{n=0}^{\infty}c_{n-1}z^n$ for small $|z|$ and $g(z)=-\sum_{n=1}^{\infty}c_{-n-1}z^{-n}$ for large $|z|$. Since

$$(-1)^n\frac{H_n^{(m-n)}}{H_n^{(m-n-1)}}z^nQ_n^{(m)}(z)=\frac{1}{H_n^{(m-n-1)}}\begin{vmatrix} c_{m-n-1} & \cdots & c_{m-1}\\ \cdots & \cdots & \cdots\\ c_{m-2} & \cdots & c_{m+n-2}\\ 1 & \cdots & z^n\end{vmatrix},$$

it follows that $(-1)^n(H_n^{(m-n)}/H_n^{(m-n-1)})\,Q_n^{(m)}(z^{-1})$ is just the denominator of the $((n-1)/n;m)$ two point Padé approximant to the function g, provided that $-n\leqslant m\leqslant n$ and that this approximant to g is normal.

Remark 1.3. $R_n^{(n)}$ is the ordinary $((n-1)/n)$ Padé approximant to f.

Remark 1.4. It can be shown that the Laurent polynomials $P_n^{(m)}$ and $Q_n^{(m)}$ satisfy the following two finite difference equations of the first order:

$$P_{n+1}^{(m)}(z)=zP_n^{(m)}(z)+\beta_nz^nQ_n^{(m)}(z)\tag{1.8}$$

with

$$\beta_n = -\frac{\Omega(z^m P_n^{(m)}(z))}{\Omega(z^{m-1} P_n^{(m)}(z)\, Q_n^{(m)}(z))}, \qquad n = 0, 1, 2,...$$

and

$$Q_{n+1}^{(m)}(z) = z^{-1} Q_n^{(m)}(z) + \delta_n z^{-n} P_n^{(m)}(z), \tag{1.9}$$

$$\delta_n = -\frac{\Omega(z^{m-2} Q_n^{(m)}(z))}{\Omega(z^{m-1} P_n^{(m)}(z)\, Q_n^{(m)}(z))}, \qquad n = 0, 1, 2,....$$

Elimination of $P_n^{(m)}$, respectively $Q_n^{(m)}$, from (1.8) and (1.9) gives

$$\beta_n P_{n+2}^{(m)}(z) - (\beta_n z + \beta_{n+1})\, P_{n+1}^{(m)}(z) + \beta_{n+1}(1 - \beta_n \delta_n)\, z P_n^{(m)}(z) = 0, \qquad n = 0, 1, 2,... \tag{1.10}$$

and

$$\delta_n Q_{n+2}^{(m)}(z) - (\delta_n z^{-1} + \delta_{n+1})\, Q_{n+1}^{(m)}(z) + \delta_{n+1}(1 - \delta_n \beta_n)\, z^{-1} Q_n^{(m)}(z) = 0, \qquad n = 0, 1, 2,.... \tag{1.11}$$

Suppose that $(c_n)_{n \in \mathbf{Z}}$ is m-normal. Then by (1.10) the denominators $V_n^{(m)}$ of $R_n^{(m)}$ satisfy

$$\beta_n V_{n+2}^{(m)}(z) - (\beta_n + \beta_{n+1} z)\, V_{n+1}^{(m)}(z) + \beta_{n+1}(1 - \beta_n \delta_n)\, z V_n^{(m)}(z) = 0, \qquad n \geqslant |m|. \tag{1.12}$$

Using (1.3) and (1.4) we get for the numerators $U_{n-1}^{(m)}$ of $R_n^{(m)}$

$$\beta_n U_{n+1}^{(m)}(z) - (\beta_n + \beta_{n+1} z)\, U_n^{(m)}(z) + \beta_{n+1}(1 - \beta_n \delta_n)\, z U_{n-1}^{(m)}(z) = 0, \qquad n \geqslant |m|. \tag{1.13}$$

It follows from (1.12) and (1.13) that there exists a T-fraction of which the nth approximant coincides with $R_n^{(m)}$ if $n \geqslant |m|$. (For the definition and elementary properties of T-fraction see [3, pp. 173–179, "Kettenbrüchen von Thron"].)

2. In this section we consider the following "operator moment problem":

> Given a sequence $(\gamma_n)_{n \in \mathbf{Z}}$ of complex numbers with $\gamma_0 = 1$, can we find a sequence $(v_n)_{n \in \mathbf{Z}}$ in the separable Hilbert space and a bounded linear operator A in H such that $A v_n = v_{n+1}$ and $\langle v_n, v_0 \rangle = \gamma_n$ for all $n \in \mathbf{Z}$?

In this paper H is a separable Hilbert space and $(e_n)_{n=0}^{\infty}$ is an orthonormal basis of H.

The proof of the following theorem is about the same as that of Theorem 4.1 of [2].

THEOREM 2.1. *Let $(\gamma_n)_{n\in\mathbf{Z}}$ be a sequence of complex numbers with $\gamma_0 \in \mathbf{R}$, $\gamma_0 > 0$. Then the following are equivalent*:

(a) $\limsup_{n\to\infty} |\gamma_n|^{1/n} < \infty$.

(b) *There exist a sequence $(v_n)_{n\in\mathbf{Z}}$ in H and a bounded linear operator A in H such that $Av_n = v_{n+1}$ and $\langle v_n, v_0\rangle = \gamma_n$ for all $n \in \mathbf{Z}$.*

Proof. (b) $\Rightarrow$ (a) is obvious.

(a) $\Rightarrow$ (b). We may assume that $\gamma_0 = 1$. Since $\limsup_{n\to\infty} |\gamma_n|^{1/n} < \infty$, there is $M > 0$ such that $|\gamma_n| \leqslant M^n$ for $n = 0, 1, 2,\ldots$. Let $\alpha_n = ((n^2+1)M^{2n} - |\gamma_n|^2)^{1/2}$, $n = 1, 2,\ldots$. Then $\alpha_n > 0$ and $n^2M^{2n} \leqslant \alpha_n^2 \leqslant (n^2+1)M^{2n}$, $n = 1, 2,\ldots$. Hence

$$\sum_{k=1}^{\infty} \left|\frac{\gamma_{n+k}}{\alpha_k}\right|^2 < \infty \qquad \text{for each } n \in \mathbf{Z} \tag{2.1}$$

and

$$\left(\frac{\alpha_{n+1}}{\alpha_n}\right)_{n=1}^{\infty} \text{ is bounded.} \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} Te_0 &= \bar{\gamma}_1 e_0 + \alpha_1 e_1 \\ \text{and} \quad Te_n &= \frac{\bar{\gamma}_0\bar{\gamma}_{n+1} - \bar{\gamma}_1\bar{\gamma}_n}{\alpha_n} e_0 - \frac{\alpha_1\bar{\gamma}_n}{\alpha_n} e_1 + \frac{\alpha_{n+1}}{\alpha_n} e_{n+1}, \qquad n = 1, 2,\ldots, \end{aligned} \tag{2.3}$$

defines a bounded linear operator T in H. Furthermore (2.3) implies

$$\begin{aligned} T^n e_0 &= \bar{\gamma}_n e_0 + \alpha_n e_n, \qquad n = 1, 2,\ldots \\ \text{and} \quad \langle T^n e_0, e_0\rangle &= \bar{\gamma}_n, \qquad n = 0, 1, 2,\ldots. \end{aligned} \tag{2.4}$$

Now, let $A = T^*$ and put

$$\begin{aligned} v_n &= A^n e_0, \qquad n = 0, 1, 2,\ldots \\ \text{and} \quad v_{-n} &= \gamma_{-n} e_0 + \sum_{k=1}^{\infty} \frac{\gamma_0\gamma_{-n+k} - \gamma_{-n}\gamma_k}{\alpha_k} e_k \qquad \text{for} \quad n = 1, 2,\ldots. \end{aligned} \tag{2.5}$$

Notice that v_{-n} is well defined by (2.1). Moreover (2.4) and (2.5) imply that

$\langle v_n, v_0 \rangle = \gamma_n$ for all $n \in \mathbf{Z}$ and it is easily verified that $Av_n = v_{n+1}$ for all $n \in \mathbf{Z}$. ∎

Remark 2.1. Let $A, T, (\gamma_n)_{n \in \mathbf{Z}}$ and $(v_n)_{n \in \mathbf{Z}}$ be as in the above proof and put $u_n = T^n e_0$, $n = 0, 1, 2, \ldots$. Assume that $\gamma_n = \bar{c}_n$ where $(c_n)_{n \in \mathbf{Z}}$ is m-seminormal. If

$$\phi_0 = u_0 \quad \text{and} \quad \phi_n = \frac{1}{H_n^{(m-n)}} \begin{vmatrix} c_{m-n} & \cdots & c_m \\ \cdots & \cdots & \cdots \\ c_{m-1} & \cdots & c_{m+n-1} \\ u_0 & \cdots & u_n \end{vmatrix}, \qquad n = 1, 2, \ldots$$

and

$$\psi_0 = v_{m-1} \quad \text{and} \quad \psi_n = \frac{(-1)^n}{\bar{H}_n^{(m-n)}} \begin{vmatrix} \bar{c}_{m-n-1} & \cdots & \bar{c}_{m-1} \\ \cdots & \cdots & \cdots \\ \bar{c}_{m-2} & \cdots & \bar{c}_{m+n-2} \\ v_{m-n-1} & \cdots & v_{m-1} \end{vmatrix}, \qquad n = 1, 2, \ldots,$$

then $\{\phi_n ; \psi_n\}_{n=0}^{\infty}$ is a biorthogonal system in H. Clearly, $\phi_n = P_n^{(m)}(T)\, u_0$, $n = 0, 1, 2, \ldots$, but since T^{-1} does not necessarily exist, we cannot say that $\psi_n = [T^{m-1} Q_n^{(m)}(T)]^* u_0$. However, in the case that there exists a function ϕ with $\phi(z) = \sum_{n=0}^{\infty} c_n z^n$ in a neighborhood of 0 and $\phi(z) = -\sum_{n=1}^{\infty} c_{-n} z^{-n}$ in a neighborhood of ∞ which is meromorphic in $\mathbf{C}^* \setminus \{p\}$, $p \neq 0$, $p \neq \infty$, we can generate the moments c_n, $n \in \mathbf{Z}$, by a linear isomorphism of H, as we will see.

Remark 2.2. For use in the proof of the next theorem we quote [2, Theorem 4.2]. Let $\sum_{n=0}^{\infty} \gamma_n z^n$ have a positive radius of convergence and let $\gamma_0 = 1$. Then the following are equivalent:

(a) There exists a compact linear operator A in H such that

$$\langle A^n e_0, e_0 \rangle = \gamma_n, \qquad n = 0, 1, 2, \ldots.$$

(b) There is a meromorphic function ϕ on $\mathbf{C}$ such that $\phi(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ in some neighborhood of 0.

In the proof of (b) ⇒ (a) of this theorem the function ϕ was written as $\phi(z) = (1 + zh(z))/(1 - zg(z))$, where g and h were entire functions. Using the power-series developments of g and h about 0, a compact operator A was constructed such that $\langle (I - zA)^{-1} e_0, e_0 \rangle = \phi(z)$ for $z \in \mathbf{C} \setminus \{\text{poles}\}$. It can be shown by elementary linear algebra that for this operator A we have:

> z is a regular value for A (i.e., $(I - zA)^{-1}$ exists as a bounded linear operator defined on all of H) if and only if $1 - zg(z) \neq 0$.

So we have the following:

LEMMA 2.1. *If ϕ is meromorphic on* $\mathbf{C}$, $\phi(0)=1$ *and ϕ does not have a pole at* $z_0 \in \mathbf{C}$, *then there exists a compact linear operator A in H such that* $\langle (I-zA)^{-1} e_0, e_0\rangle = \phi(z)$ *for* $z \in \mathbf{C}\setminus\{poles\}$ *and z_0 is a regular value for A.*

THEOREM 2.2. *Let* $(\gamma_n)_{n\in\mathbf{Z}}$ *be a sequence of complex numbers with* $\gamma_0 = 1$ *and let* $p \in \mathbf{C}$, $p \neq 0$. *Then the following are equivalent*:

(a) *There exists a meromorphic function ϕ on* $\mathbf{C}^*\setminus\{p\}$ *with* $\phi(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ *in some neighborhood of* 0 *and* $\phi(z) = -\sum_{n=1}^{\infty} \gamma_{-n} z^{-n}$ *in some neoghborhood of* ∞.

(b) *There exists a compact linear operator A in H such that* $\langle [p^{-1}(I+A)]^n e_0, e_0\rangle = \gamma_n$ *for all* $n \in \mathbf{Z}$.

Proof. (a) $\Rightarrow$ (b). If $h(z) = (1+z)^{-1}\phi(pz(1+z)^{-1})$, then h is meromorphic on $\mathbf{C}$, $h(0)=1$ and -1 is not a pole of h. By Lemma 2.1 there exists a compact linear operator A in H such that $h(z) = \langle (I-zA)^{-1} e_0, e_0\rangle$, $z \in \mathbf{C}\setminus\{\text{poles}\}$ and $(I+A)^{-1}$ exists. Clearly $\phi(z) = p(p-z)^{-1} h(z(p-z)^{-1})$, so it follows from

$$p(p-z)^{-1}[I - z(p-z)^{-1} A]^{-1} = [I - zp^{-1}(I+A)]^{-1} \tag{2.6}$$

that $\phi(z) = \langle [I - zp^{-1}(I+A)]^{-1} e_0, e_0\rangle$. Hence for small $|z|$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} \gamma_n z^n &= \phi(z) = \langle [I - zp^{-1}(I+A)]^{-1} e_0, e_0\rangle \\ &= \left\langle \sum_{n=0}^{\infty} z^n [p^{-1}(I+A)]^n e_0, e_0 \right\rangle = \sum_{n=0}^{\infty} z^n \langle [p^{-1}(I+A)]^n e_0, e_0\rangle,\end{aligned}$$

so $\gamma_n = \langle [p^{-1}(I+A)]^n e_0, e_0\rangle$ for $n = 0, 1, 2, \ldots$. Since $(I+A)^{-1}$ exists we have

$$[I - zp^{-1}(I+A)]^{-1} = -z^{-1}p(I+A)^{-1}[I - z^{-1}p(I+A)^{-1}]^{-1} \tag{2.7}$$

if zp^{-1} is regular for $I+A$. For sufficiently large $|z|$ this gives

$$\begin{aligned}-\sum_{n=1}^{\infty} \gamma_{-n} z^{-n} &= \phi(z) = \langle -z^{-1}p(I+A)^{-1}[I - z^{-1}p(I+A)^{-1}]^{-1} e_0, e_0\rangle \\ &= \left\langle -\sum_{n=1}^{\infty} z^{-n}[p(I+A)^{-1}]^n e_0, e_0 \right\rangle \\ &= -\sum_{n=1}^{\infty} z^{-n} \langle [p^{-1}(I+A)]^{-n} e_0, e_0\rangle,\end{aligned}$$

hence $\gamma_{-n} = \langle [p^{-1}(I+A)]^{-n} e_0, e_0\rangle$ for $n = 1, 2, \ldots$.

(b) ⇒ (a). Since A is compact, $h(z)=\langle[I-zA]^{-1}e_0, e_0\rangle$ is meromorphic in $\mathbf{C}$, so $\phi(z)=p(p-z)^{-1}h(z(p-z)^{-1})$ is meromorphic in $\mathbf{C}^*\backslash\{p\}$. Using (2.6), (2.7) and the fact that $(I+A)^{-1}$ exists, we get

$$\phi(z)=\sum_{n=0}^{\infty}\gamma_n z^n \qquad \text{for small } |z|$$

and

$$\phi(z)=-\sum_{n=1}^{\infty}\gamma_{-n}z^{-n} \qquad \text{for large } |z|. \quad ∎$$

Remark 2.3. Let $(c_n)_{n\in\mathbf{Z}}$ and f be as in Section 1 and assume that f is meromorphic in $\mathbf{C}^*\backslash\{p\}$, $p\neq 0$, $p\neq\infty$. Then by the above theorem there exists a linear isomorphism T in H such that $\langle T^n e_0, e_0\rangle = c_n$ for all $n\in\mathbf{Z}$. If $P_n^{(m)}$ and $Q_n^{(m)}$ are as in Section 1, then it follows from Remark 2.1 that $\{P_n^{(m)}(T)\,e_0;\,[T^{m-1}Q_n^{(m)}(T)]^*\,e_0\}_{n=0}^{\infty}$ is a biorthogonal system in H.

3. We now return to the function f of Section 1 with $f(0)=1$ and $f(z)=\sum_{n=0}^{\infty}c_n z^n$ and $f(z)=-\sum_{n=1}^{\infty}c_{-n}z^{-n}$ in neighborhoods of 0, respectively ∞. We assume that the sequence $(c_n)_{n\in\mathbf{Z}}$ is m-seminormal for some nonnegative integer m.

If $n\geqslant m$ and $R_n^{(m)}$ has the form (1.5) with $b_0=1$, then $a_0,\dots,a_{n-1}$, $b_0,\dots,b_n$ is the unique solution with $b_0=1$ of the systems of linear equations given by (1.3) and (1.4). Since $m\geqslant 0$ these systems are

$$\begin{aligned}
a_0 &= c_0 b_0,\\
a_1 &= c_1 b_0 + c_0 b_1,\\
&\cdots\cdots\cdots\cdots\\
a_{n-1} &= c_{n-1} b_0 + c_{n-2} b_1 + \cdots + c_0 b_{n-1},\\
0 &= c_n b_0 + c_{n-1} b_1 + \cdots + c_1 b_{n-1} + c_0 b_n,\\
&\cdots\cdots\cdots\cdots\\
0 &= c_{m+n-1} b_0 + c_{m+n-2} b_1 + \cdots + c_m b_{n-1} + c_{m-1} b_n
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
-a_m &= c_{-1} b_{m+1} + c_{-2} b_{m+2} + \cdots + c_{m-n+1} b_{n-1} + c_{m-n} b_n,\\
-a_{m+1} &= c_{-1} b_{m+2} + \cdots + c_{m-n+2} b_{n-1} + c_{m-n+1} b_n,\\
&\cdots\cdots\cdots\cdots\\
-a_{n-2} &= c_{-1} b_{n-1} + c_{-2} b_n,\\
-a_{n-1} &= c_{-1} b_n.
\end{aligned} \tag{3.2}$$

By Theorem 2.1 and Remark 2.1 there are sequences $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n\in\mathbf{Z}}$ with $u_0 = v_0 = e_0$ in H and a bounded linear operator $T: H \to H$ such that

$$T^n e_0 = u_n, \qquad n = 0, 1, 2,..., \qquad \text{and} \qquad T^* v_n = v_{n+1}, \qquad n \in \mathbf{Z}, \quad (3.3)$$

and

$$\langle u_n, v_k \rangle = c_{n+k}, \qquad n = 0, 1, 2,..., \quad k \in \mathbf{Z}. \qquad (3.4)$$

For every $n \in \mathbf{N}$ we put

$$U_n = \operatorname{span}\{u_0, u_1,..., u_{n-1}\}$$

and

$$V_n = \operatorname{span}\{v_{m-n}, v_{m-n+1},..., v_{m-1}\}.$$

It follows from the normality of $(c_n)_{n\in\mathbf{Z}}$ that $(u_n)_{n=0}^{\infty}$ and $(v_{m-n})_{n=1}^{\infty}$ are both independent sequences in H and that

$$U_n \cap V_n^{\perp} = \{0\} \qquad \text{and} \qquad U_n^{\perp} \cap V_n = \{0\}, \qquad n = 1, 2,....$$

Since $\dim U_n < \infty$ and $V_n^{\perp}$ is closed, this implies

$$H = U_n \oplus V_n^{\perp}, \qquad n = 1, 2,....$$

Let $E_n: H \to H$ be the continous linear projection onto U_n with kernel $V_n^{\perp}$, $n = 1, 2,...$, and let $T_n: H \to H$ be defined by $T_n = E_n T E_n$, $n = 1, 2,...$. Then clearly $T_n(H) \subset U_n$ and by (3.3)

$$T_n^k u_0 = u_k \qquad \text{for} \quad k = 0, 1,..., n-1. \qquad (3.5)$$

Since $u_n - P_n^{(m)}(T)\, u_0 \in U_n$ and $P_n^{(m)}(T)\, u_0 \in V_n^{\perp}$ (cf. the biorthogonal system $\{\phi_n; \psi_n\}_{n=0}^{\infty}$ in Remark 2.1), we have

$$E_n u_n = u_n - P_n^{(m)}(T)\, u_0, \qquad n = 1, 2,...,$$

hence

$$T_n^n u_0 = T_n u_{n-1} = E_n T E_n u_{n-1} = E_n T u_{n-1} = E_n u_n = u_n - P_n^{(m)}(T)\, u_0, \qquad n = 1, 2,...,$$

and

$$P_n^{(m)}(T_n)\, u_0 = 0, \qquad n = 1, 2,.... \qquad (3.6)$$

This implies that $P_n^{(m)}(T_n)\, x = 0$ for all $x \in U_n$ so T_n satisfies the polynomial equation

$$T_n P_n^{(m)}(T_n) = 0, \qquad n = 1, 2,.... \qquad (3.7)$$

If $\bar{T}_n$ denotes the restriction of T_n to U_n, then it is obvious from the matrix representation of $\bar{T}_n$ with respect to the basis $u_0, u_1, \ldots, u_{n-1}$ of U_n that $P_n^{(m)}$ is the characteristic polynomial of $\bar{T}_n$ and that $\bar{T}_n$ is an isomorphism of U_n if $P_n^{(m)}(0) \neq 0$ which holds if $(c_n)_{n\in\mathbf{Z}}$ is m-seminormal.

THEOREM 3.1. *For $n \geqslant m$ and $z \in \mathbf{C}\setminus\{$poles of $R_n^{(m)}\}$ we have*

$$\langle (I - zT_n)^{-1} u_0, u_0\rangle = R_n^{(m)}(z).$$

Proof. Let $n \geqslant m$ and let $P_n^{(m)}(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n$ with $b_0 = 1$. Then by (3.7) we have

$$b_0 T_n^{n+k+1} + b_1 T_n^{n+k} + \cdots + b_{n-1} T_n^{k+2} + b_n T_n^{k+1} = 0 \qquad \text{for} \quad k = 0, 1, 2, \ldots. \tag{3.8}$$

Since T_n is compact, $(I - zT_n)^{-1}$ is an operator-valued meromorphic function on $\mathbf{C}$ which satisfies

$$(I - zT_n)^{-1} = \sum_{k=0}^{\infty} z^k T_n^k \qquad \text{for sufficiently small } |z|.$$

Using (3.8) we get for small $|z|$

$$z^n P_n^{(m)}(z^{-1})(I - zT_n)^{-1} = B_0 + zB_1 + \cdots + z^n B_n,$$

where

$$B_j = b_0 T_n^j + b_1 T_n^{j-1} + \cdots + b_j I, \qquad j = 0, 1, \ldots, n. \tag{3.9}$$

Hence for small $|z|$

$$(I - zT_n)^{-1} = \frac{B_0 + zB_1 + \cdots + z^n B_n}{z^n P_n^{(m)}(z^{-1})}. \tag{3.10}$$

Since both sides of (3.10) are meromorphic on $\mathbf{C}$, (3.10) holds for all $z \in \mathbf{C}\setminus\{\text{poles}\}$. If we take $j = n$ in (3.9) we get $B_n = P_n^{(m)}(T_n)$, so by (3.6) we have

$$B_n u_0 = 0. \tag{3.11}$$

It follows from (3.9), (3.5), (3.4) and (3.1) that

$$\langle B_j u_0, u_0\rangle = a_j, \qquad j = 0, 1, \ldots, n-1. \tag{3.12}$$

Now (3.11) and (3.12) yield

$$\langle (I - zT_n)^{-1} u_0, u_0\rangle = R_n^{(m)} \qquad \text{for} \quad z \in \mathbf{C}\setminus\{\text{poles}\}. \quad \blacksquare$$

Remark 3.1. It follows almost immediately from [1, VII.3.16, "Minimal equation theorem"] that an operator T in H satisfies a non-trivial polynomial equation $P(T)=0$ if and only if the spectrum of T consists only of a finite set of poles of $(\lambda I - T)^{-1}$ [1, VII.5.17].

In order to get convergence results for the sequence of approximants $(R_n^{(m)}(z))_{n=m}^{\infty}$ to f we assume from now on that f is meromorphic on $\mathbf{C}^* \setminus \{p\}$, $p \neq 0$, $p \neq \infty$. Then by Theorem 2.2 there exists a compact linear operator A in H such that $\langle [p^{-1}(I+A)]^k e_0, e_0 \rangle = c_k$ for all $k \in \mathbf{Z}$. Put

$$T = p^{-1}(I+A) \tag{3.13}$$

and let $T^k e_0 = u_k$ and $(T^*)^k e_0 = v_k$ for all $k \in \mathbf{Z}$ and define the subspaces U_n and V_n, the projections E_n and the operators T_n, $n = 1, 2, ...$, as in the beginning of this section. We also assume that the biorthogonal system

$$\left\{ P_n^{(m)}(T)\, u_0 ;\ \left[\frac{(-1)^n H_n^{(m-n)}}{H_{n+1}^{(m-n-1)}} T^{m-1} Q_n^{(m)}(T) \right]^* u_0 \right\}_{n=0}^{\infty}$$

is a Schauder basis of H together with the associated sequence of coefficient functionals. It follows from elementary theory of bases in Banach spaces that the assumption that $(P_n^{(m)}(T)\, u_0)_{n=0}^{\infty}$ is a basis of H is equivalent to

$$H = \overline{\operatorname{span}\{u_n\}_{n=0}^{\infty}} \text{ and } (\|E_n\|)_{n=1}^{\infty} \text{ is bounded,}$$

and that this assumption is also equivalent to

$$\lim_{n \to \infty} E_n x = x \qquad \text{for all } x \in H.$$

See for instance [4, Chap. I, Theorem 4.1].

Since $A = pT - I$

$$\operatorname{span}\{u_0, Au_0, ..., A^{n-1}u_0\} = U_n, \qquad n = 1, 2, ... \tag{3.14}$$

Let the linear operators A_n be defined by

$$A_n = E_n A E_n, \qquad n = 1, 2, ... \tag{3.15}$$

By (3.14) a slight modification of Vorobyev's method [5, Chap. II] applied to the compact operators A and the operators A_n yields

LEMMA 3.1. (i) $\lim_{n \to \infty} \|A_n - A\| = 0$.

(ii) *If μ is regular for A, then μ is regular for A_n if n is sufficiently large.*

(iii) $\lim_{n \to \infty} \|(I - \mu A_n)^{-1} - (I - \mu A)^{-1}\| = 0$ *if μ is regular for A.*

(iv) $\lim_{n\to\infty} \|(I-\mu A_n)^{-1}u_0-(I-\mu A)^{-1}u_0\|^{1/n}=0$ (*i.e.*, $(I-\mu A_n)^{-1}u_0 \to (I-\mu A)^{-1}u_0$ *as* $n\to\infty$, *faster then any geometric progression*) *if* μ *is regular for* A.

Remark 3.2. The fact that $(P_n^{(m)}(T)\,u_0)_{n=0}^{\infty}$ is a basis of H implies that $([T^{m-1}Q_n^{(m)}(T)]^*\,u_0)_{n=0}^{\infty}$ is a basis of H as well.

LEMMA 3.2. *If* z *is regular for* T, *then* z *is regular for* T_n *if* n *is sufficiently large.*

Proof. It follows from (2.6) that $z(p-z)^{-1}$ is regular for A so by Lemma 3.1 there is n_0 such that $z(p-z)^{-1}$ is regular for A_n as $n\geqslant n_0$. Let $n\geqslant n_0$. Since T_n has finite dimensional range, it suffices to show that $I-zT_n$ is one-to-one. Let $x-zT_nx=0$. Then clearly $x\in U_n$ and $E_nx=x$. Since $T_n=E_nTE_n=E_np^{-1}(I+A)E_n=p^{-1}(E_n+E_nAE_n)=p^{-1}(E_n+A_n)$ by (3.15), it follows that $x-z(p-z)^{-1}A_nx=0$ and this implies $x=0$, for $z(p-z)^{-1}$ is regular for A_n. Hence $I-zT_n$ is one-to-one. ∎

LEMMA 3.3. *Let* z *be regular for* T, $x=(I-zT)^{-1}u_0$ *and* $x_n=(I-zT_n)^{-1}u_0$ *for* n *sufficiently large. Then*

$$\lim_{n\to\infty}\|x_n-x\|^{1/n}=0. \tag{3.16}$$

Proof. By (2.6) we have $x=p(p-z)^{-1}(I-z(p-z)^{-1}A)^{-1}u_0$ with $z(p-z)^{-1}$ regular for A. In a similar way, using $x_n\in U_n$, we get $x_n=p(p-z)^{-1}(I-z(p-z)^{-1}A_n)^{-1}u_0$ for large n. Hence (3.16) follows from Lemma 3.1. ∎

THEOREM 3.2. *Let* f *be meromorphic on* $\mathbf{C}^*\setminus\{p\}$, $p\neq 0$, $p\neq\infty$ *and let* $f(z)=\sum_{k=0}^{\infty}c_kz^k$ *in some neighborhood of* 0, $f(0)=1$, *and* $f(z)=-\sum_{k=1}^{\infty}c_{-k}z^{-k}$ *in some neighborhood of* ∞. *Suppose that* $(c_k)_{k\in\mathbf{Z}}$ *is* m-*seminormal for some nonnegative integer* m. *Let* T *be as in* (3.13) *and assume that* $(P_n^{(m)}(T)\,u_0)_{n=0}^{\infty}$ *is a basis of* H. *Then the sequence* $(R_n^{(m)}(z))_{n=1}^{\infty}$ *of* $((n-1)/n;m)$ *two point Padé approximants to* f *converges to* $f(z)$ *for every* z *which is regular for* T *and the convergence is faster then any geometric progression.*

Proof. If z is regular for T and n is large enough, we have by Theorem 3.1

$$R_n^{(m)}(z)=\langle (I-zT_n)^{-1}u_0,u_0\rangle$$

and because

$$f(z)=\langle (I-zT)^{-1}u_0,u_0\rangle$$

Lemma 3.3 gives

$$|R_n^{(m)}(z)-f(z)|^{1/n} \leqslant \|(I-zT_n)^{-1}u_0-(I-zT)^{-1}u_0\|^{1/n} \to 0 \quad \text{as} \quad n \to \infty. \quad \blacksquare$$

Remark 3.3. The values of z which are not regular for T form a countable set which has no accumulation point in $\mathbf{C}^*$ except possibly p.

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